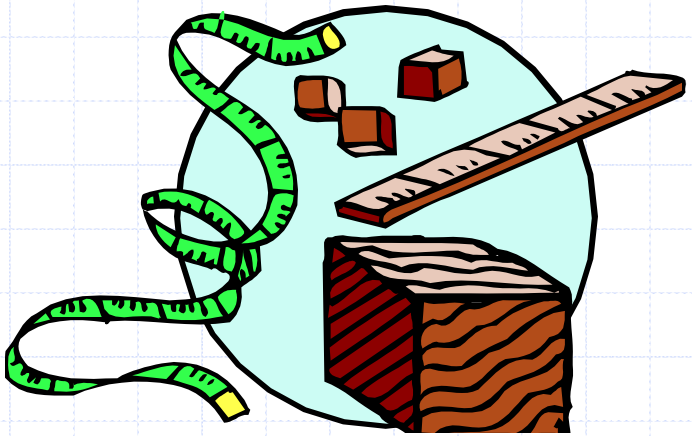
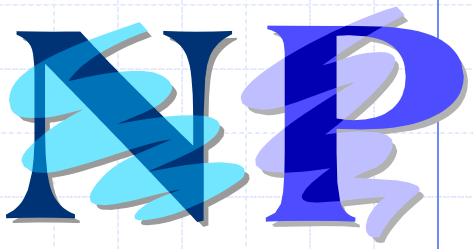
Approximation Algorithms

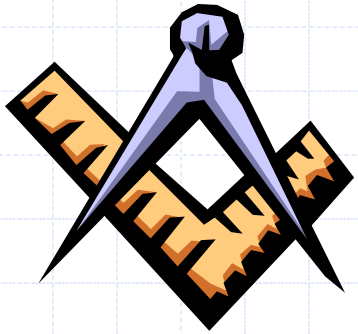


Outline and Reading



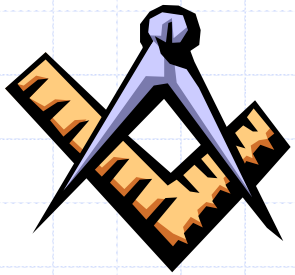
- Approximation Algorithms for NP-Complete Problems (§13.4)
  - Approximation ratios
  - Polynomial-Time Approximation Schemes (§13.4.1)
  - 2-Approximation for Vertex Cover (§13.4.2)
  - 2-Approximation for TSP special case (§13.4.3)
  - Log n-Approximation for Set Cover (§13.4.4)

Approximation Ratios



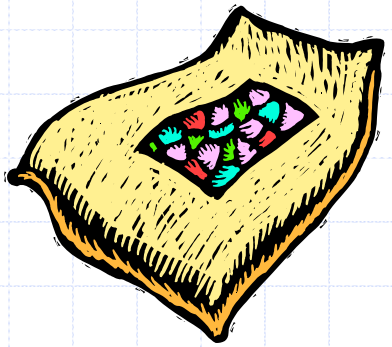
- Optimization Problems
  - We have some problem instance x that has many feasible "solutions".
  - We are trying to minimize (or maximize) some cost function c(S) for a "solution" S to x. For example,
    - Finding a minimum spanning tree of a graph
    - Finding a smallest vertex cover of a graph
    - Finding a smallest traveling salesperson tour in a graph
- An approximation produces a solution T
  - T is a **k-approximation** to the optimal solution OPT if $c(T)/c(OPT) \leq k$ (assuming a min. prob.; a maximization approximation would be the reverse)

Polynomial-Time Approximation Schemes

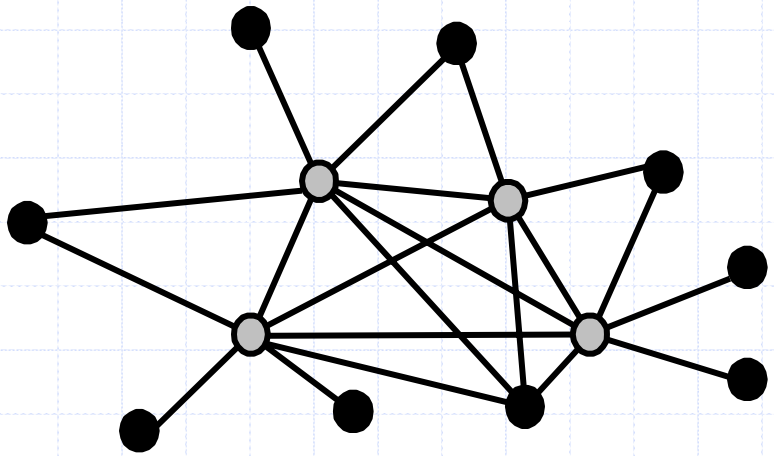


- A problem L has a **polynomial-time approximation scheme (PTAS)** if it has a polynomial-time $(1+\varepsilon)$-approximation algorithm, for any fixed $\varepsilon > 0$ (this value can appear in the running time).
- 0/1 Knapsack has a PTAS, with a running time that is $O(n^3/\varepsilon)$. Please see §13.4.1 in Goodrich-Tamassia for details.

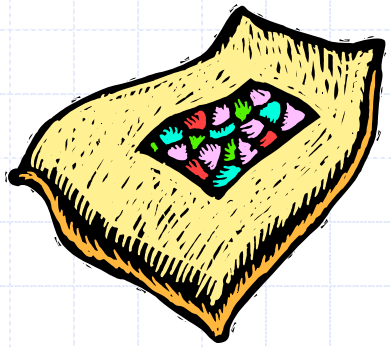
Vertex Cover



- A **vertex cover** of graph G=(V,E) is a subset W of V, such that, for every (a,b) in E, a is in W or b is in W.
- OPT-VERTEX-COVER: Given an graph G, find a vertex cover of G with smallest size.
- OPT-VERTEX-COVER is NP-hard.



A 2-Approximation for Vertex Cover

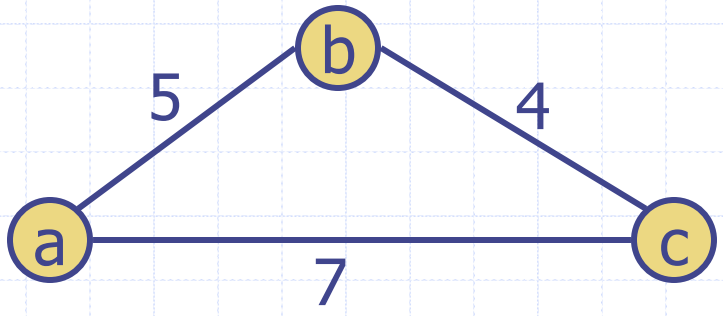


- Every chosen edge e has both ends in C
- But e must be covered by an optimal cover; hence, one end of e must be in OPT
- Thus, there is at most twice as many vertices in C as in OPT.
- That is, C is a 2-approx. of OPT
- Running time: O(m)

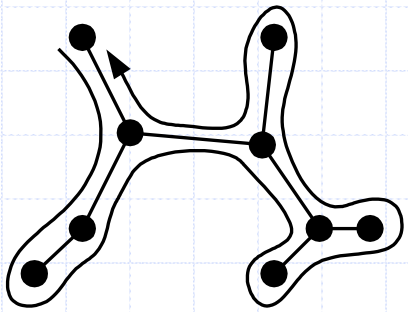
Algorithm *VertexCoverApprox*(*G*)
  Input graph *G*
  Output a vertex cover *C* for *G*
  C ← empty set
  H ← *G*
  while *H* has edges
    e ← *H.removeEdge*(*H.anEdge*())
    v ← *H.origin*(*e*)
    w ← *H.destination*(*e*)
    C.add(*v*)
    C.add(*w*)
    for each *f* incident to *v* or *w*
      H.removeEdge(*f*)
  return *C*

Special Case of the Traveling Salesperson Problem

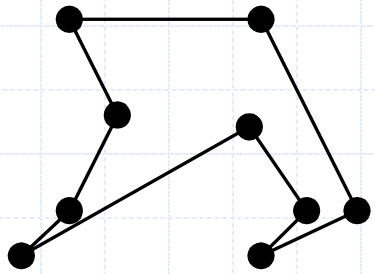
- OPT-TSP: Given a complete, weighted graph, find a cycle of minimum cost that visits each vertex.
  - OPT-TSP is NP-hard
  - Special case: edge weights satisfy the triangle inequality (which is common in many applications):
    - $w(a,b) + w(b,c) \geq w(a,c)$



A 2-Approximation for TSP Special Case



Euler tour P of MST M



Output tour T

Algorithm *TSPApprox*(*G*)

Input weighted complete graph ***G,*** satisfying the triangle inequality

Output a TSP tour ***T*** for ***G***

M ← a minimum spanning tree for ***G***

P ← an Euler tour traversal of ***M,*** starting at some vertex ***s***

T ← empty list

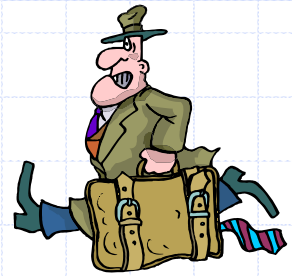
for each vertex ***v*** in ***P*** (in traversal order)

if this is ***v***'s first appearance in ***P*** **then** *T.insertLast*(*v*)

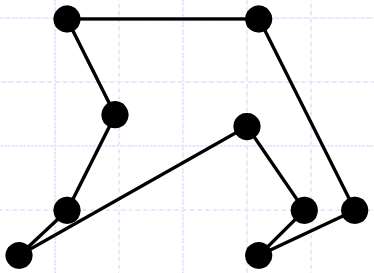
T.insertLast(*s*)

return ***T***

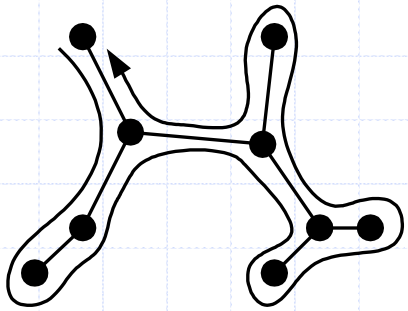
A 2-Approximation for TSP Special Case - Proof



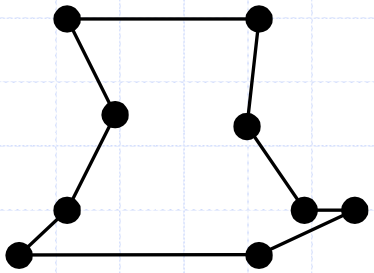
- The optimal tour is a spanning tour; hence $|M| \leq |OPT|$.
- The Euler tour P visits each edge of M twice; hence $|P|=2|M|$
- Each time we shortcut a vertex in the Euler Tour we will not increase the total length, by the triangle inequality ($w(a,b) + w(b,c) \geq w(a,c)$); hence, $|T| \leq |P|$.
- Therefore, $|T| \leq |P| = 2|M| \leq 2|OPT|$



Output tour T
(at most the cost of P)

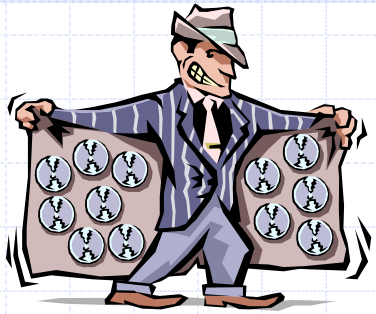


Euler tour P of MST M
(twice the cost of M)



Optimal tour OPT
(at least the cost of MST M)

Set Cover



- OPT-SET-COVER: Given a collection of m sets, find the smallest number of them whose union is the same as the whole collection of m sets?
  - OPT-SET-COVER is NP-hard
- Greedy approach produces an O(log n)-approximation algorithm. See §13.4.4 for details.

Algorithm *SetCoverApprox*(*G*)
  Input a collection of sets $S_1 \ldots S_m$
  Output a subcollection C with same union
  $F \leftarrow \{S_1, S_2, \ldots, S_m\}$
  $C \leftarrow$ empty set
  $U \leftarrow$ union of $S_1 \ldots S_m$
  while U is not empty
    $S_i \leftarrow$ set in F with most elements in U
    $F.remove(S_i)$
    $C.add(S_i)$
    Remove all elements in S_i from U
  return C