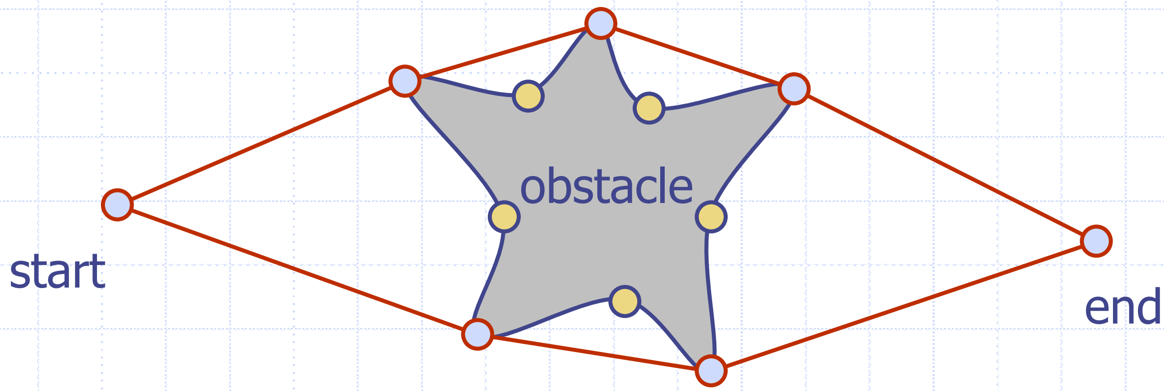
Convex Hull

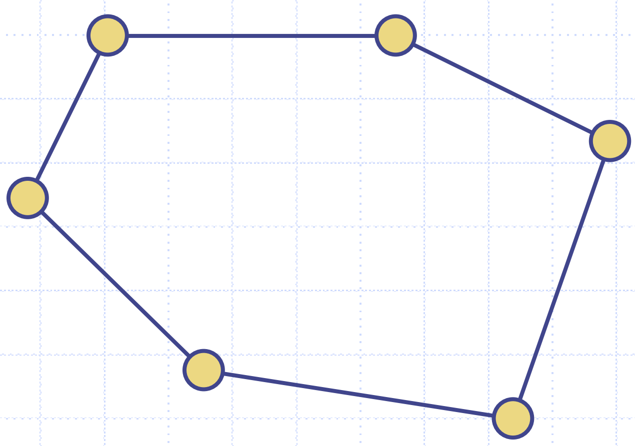


Outline and Reading

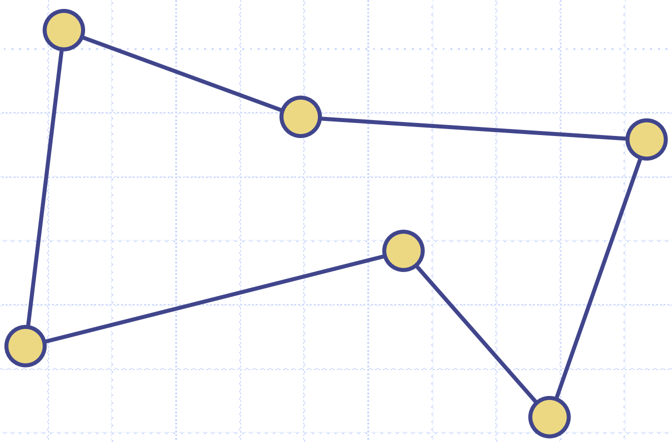
- Convex hull (§12.5.2)
- Orientation (§12.5.1-2)
- Sorting by angle (§12.5.5)
- Graham scan (§12.5.5)
- Analysis (§12.5.5)

Convex Polygon

- A convex polygon is a nonintersecting polygon whose internal angles are all convex (i.e., less than π)
- In a convex polygon, a segment joining two vertices of the polygon lies entirely inside the polygon



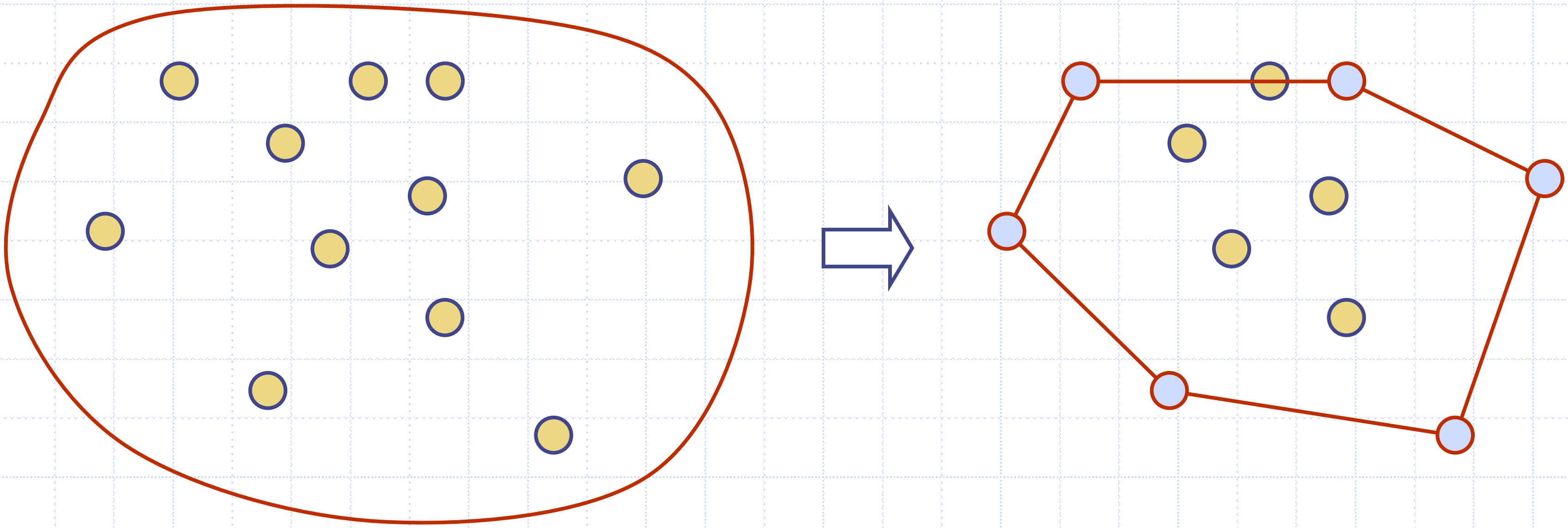
convex



nonconvex

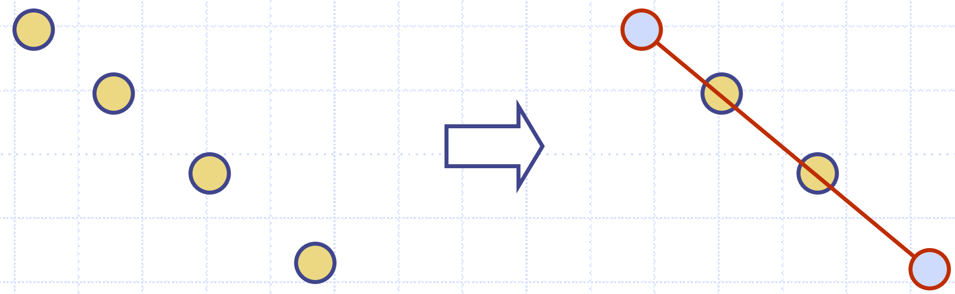
Convex Hull

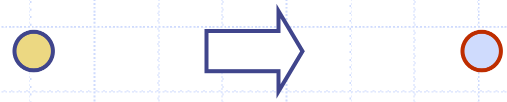
- The convex hull of a set of points is the smallest convex polygon containing the points
- Think of a rubber band snapping around the points



Special Cases

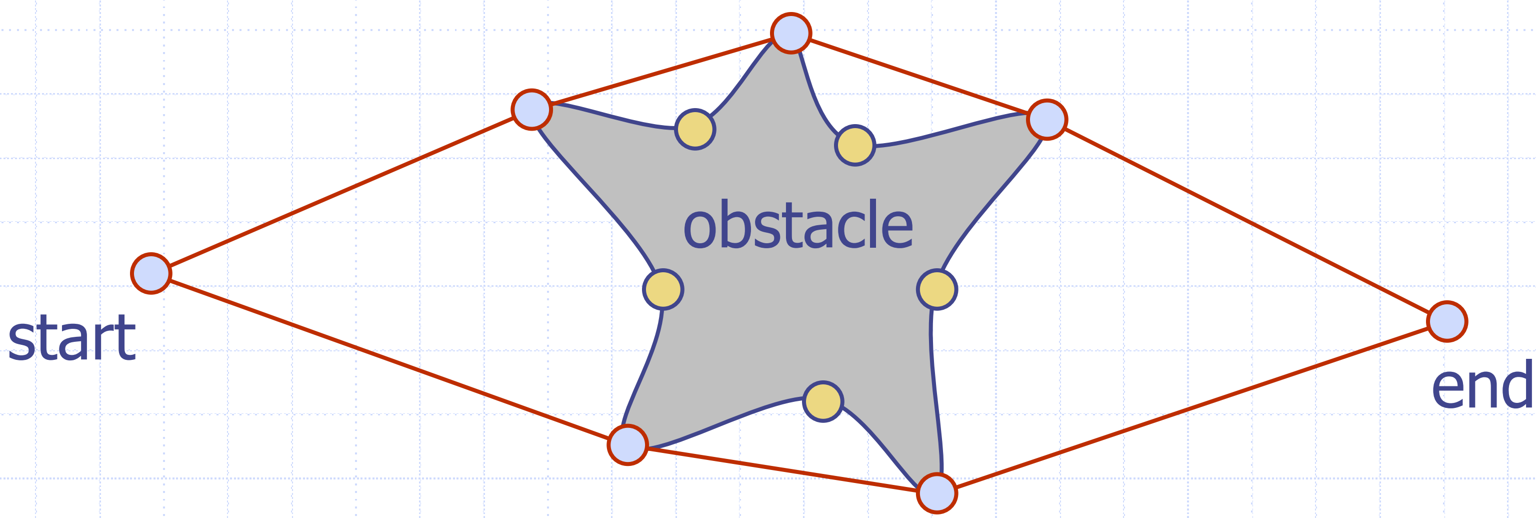
- The convex hull is a segment
  - Two points
  - All the points are collinear
- The convex hull is a point
  - there is one point
  - All the points are coincident





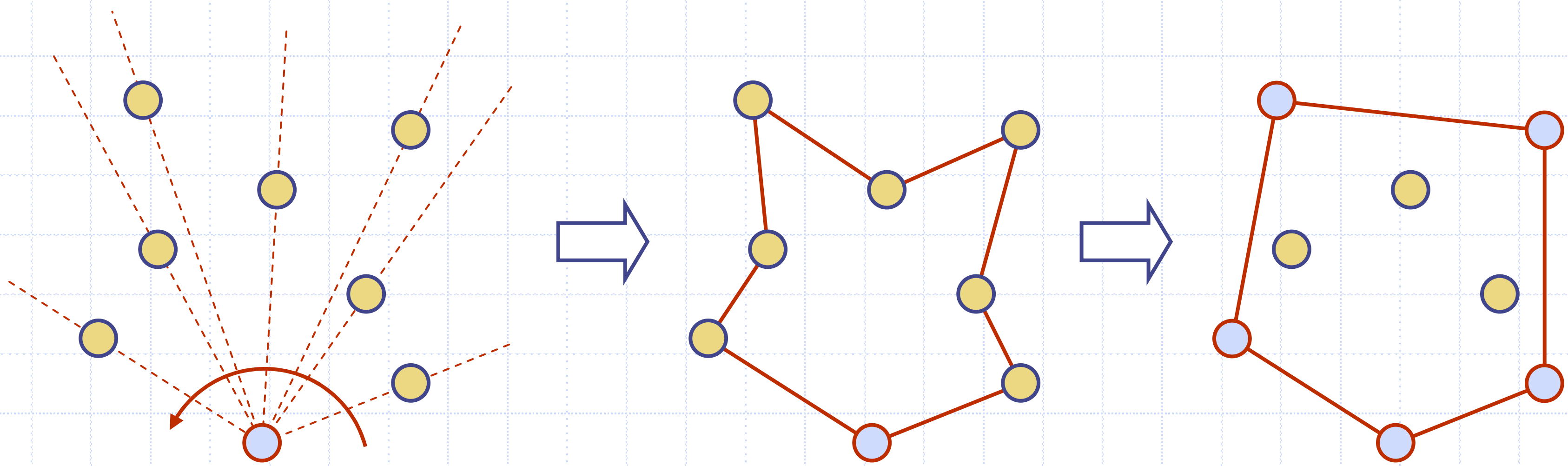
Applications

- Motion planning
  - Find an optimal route that avoids obstacles for a robot
- Geometric algorithms
  - Convex hull is like a two-dimensional sorting



Computing the Convex Hull

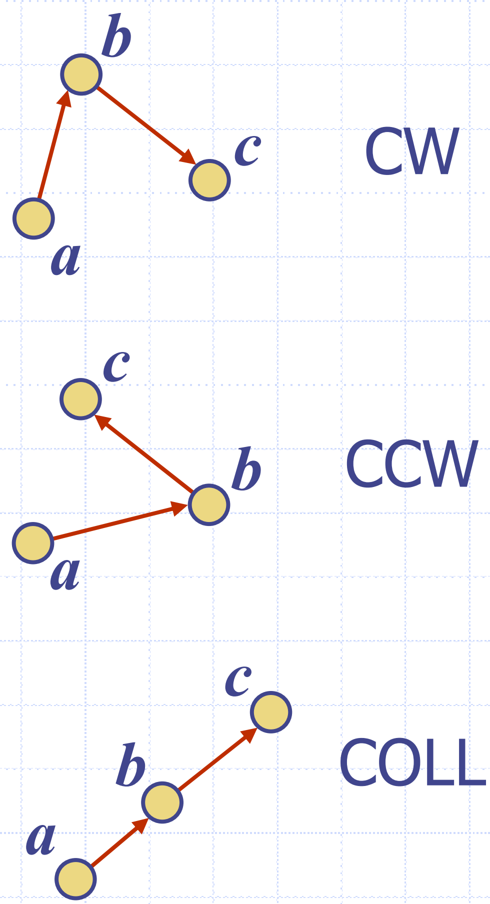
- The following method computes the convex hull of a set of points
  - Phase 1: Find the lowest point (anchor point)
  - Phase 2: Form a nonintersecting polygon by sorting the points counterclockwise around the anchor point
  - Phase 3: While the polygon has a nonconvex vertex, remove it



Orientation

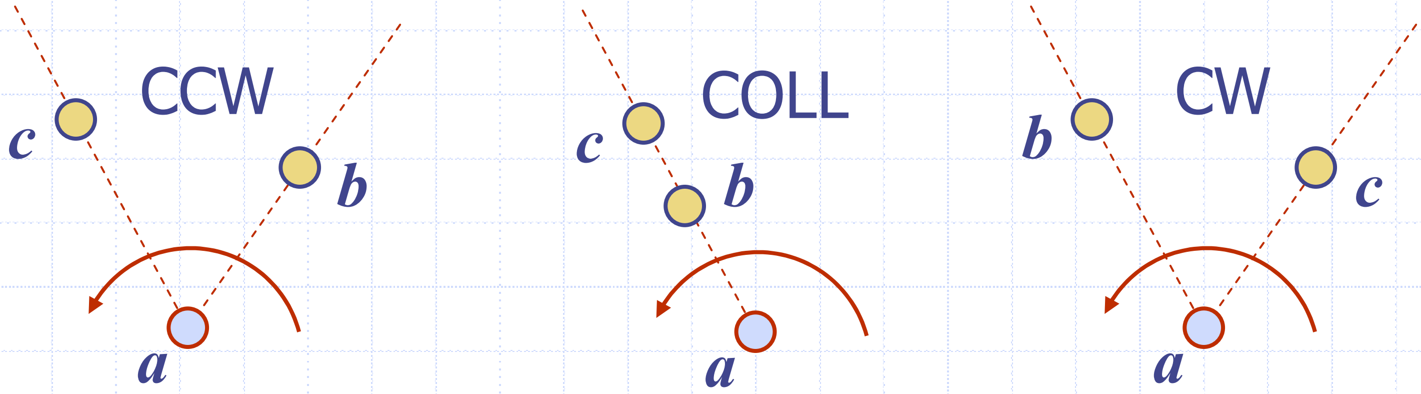
- The orientation of three points in the plane is clockwise, counterclockwise, or collinear
- ***orientation***(***a***, ***b***, ***c***)
  - clockwise (CW, right turn)
  - counterclockwise (CCW, left turn)
  - collinear (COLL, no turn)
- The orientation of three points is characterized by the sign of the determinant $\Delta(a, b, c)$, whose absolute value is twice the area of the triangle with vertices ***a***, ***b*** and ***c***

$$\Delta(a,b,c) = \begin{vmatrix} x_a & y_a & 1 \\ x_b & y_b & 1 \\ x_c & y_c & 1 \end{vmatrix}$$



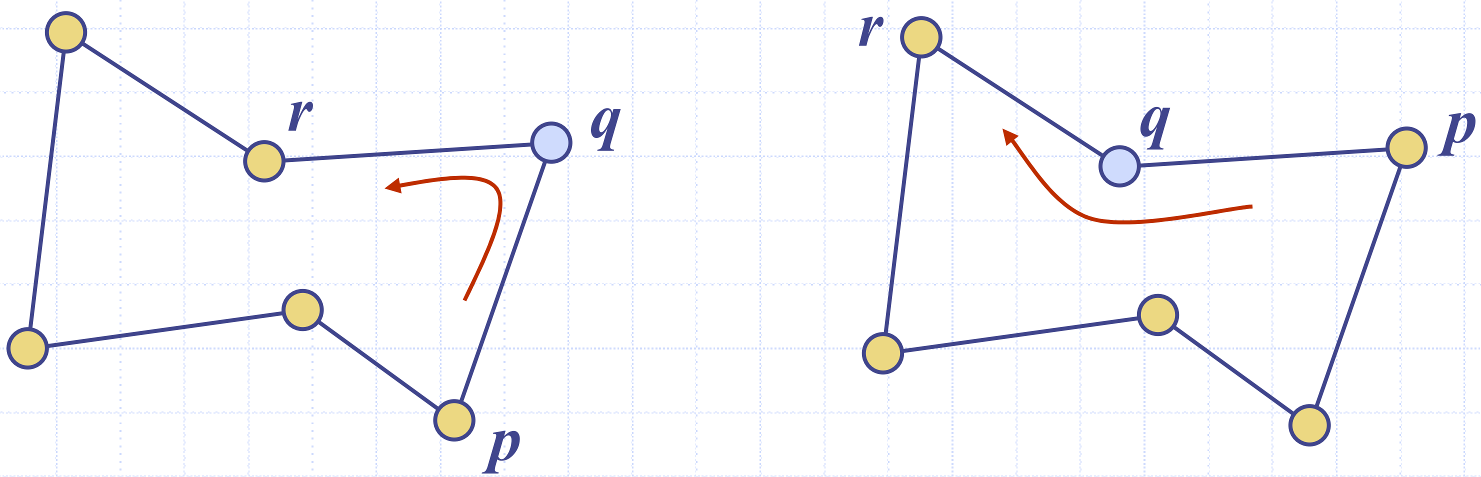
Sorting by Angle

- Computing angles from coordinates is complex and leads to numerical inaccuracy
- We can sort a set of points by angle with respect to the anchor point a using a comparator based on the orientation function
  - $b < c \Leftrightarrow orientation(a, b, c) = \text{CCW}$
  - $b = c \Leftrightarrow orientation(a, b, c) = \text{COLL}$
  - $b > c \Leftrightarrow orientation(a, b, c) = \text{CW}$



Removing Nonconvex Vertices

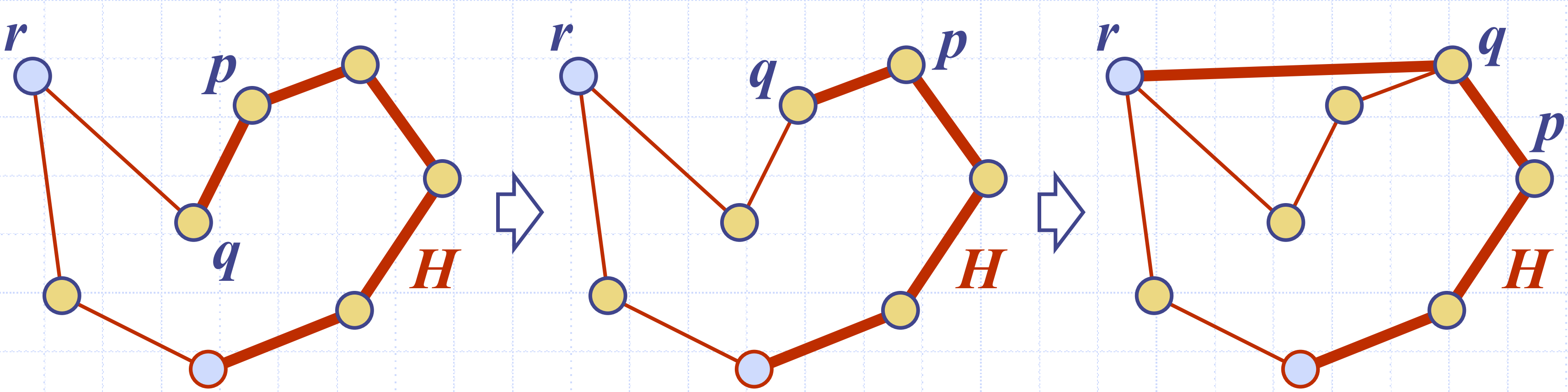
- Testing whether a vertex is convex can be done using the orientation function
- Let p, q and r be three consecutive vertices of a polygon, in counterclockwise order
  - q convex $\Leftrightarrow$ ***orientation***$(p, q, r) =$ CCW
  - q nonconvex $\Leftrightarrow$ ***orientation***$(p, q, r) =$ CW or COLL



Graham Scan

- The Graham scan is a systematic procedure for removing nonconvex vertices from a polygon
- The polygon is traversed counterclockwise and a sequence H of vertices is maintained

for each vertex r of the polygon
  Let q and p be the last and second last vertex of H
  while *orientation*(p, q, r) = CW or COLL
    remove q from H
    $q \leftarrow p$
    $p \leftarrow$ vertex preceding p in H
  Add r to the end of H



Analysis

- Computing the convex hull of a set of points takes $O(n \log n)$ time
  - Finding the anchor point takes $O(n)$ time
  - Sorting the points counterclockwise around the anchor point takes $O(n \log n)$ time
    - Use the orientation comparator and any sorting algorithm that runs in $O(n \log n)$ time (e.g., heap-sort or merge-sort)
  - The Graham scan takes $O(n)$ time
    - Each point is inserted once in sequence H
    - Each vertex is removed at most once from sequence H