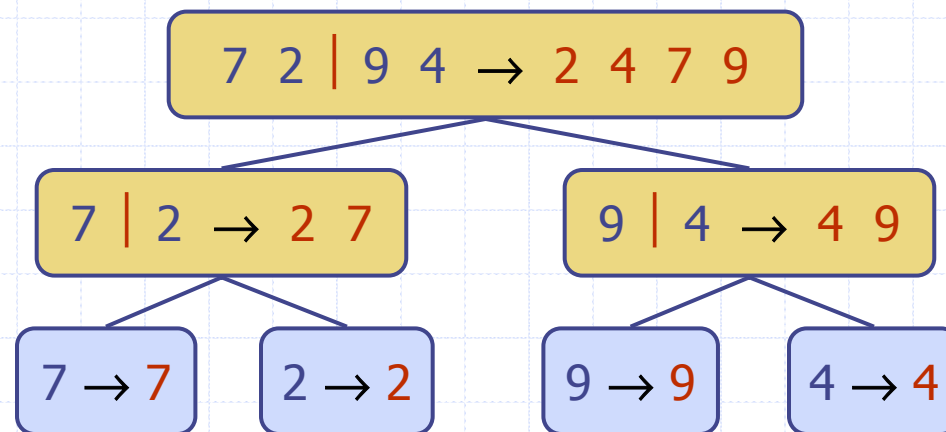


Divide-and-Conquer

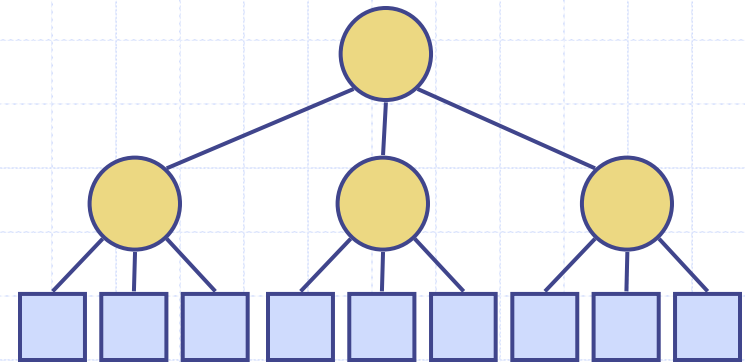


Outline and Reading

- ◆ Divide-and-conquer paradigm (§5.2)
- ◆ Review Merge-sort (§4.1.1)
- ◆ Recurrence Equations (§5.2.1)
 - Iterative substitution
 - Recursion trees
 - Guess-and-test
 - The master method
- ◆ Integer Multiplication (§5.2.2)

Divide-and-Conquer

- ◆ **Divide-and conquer** is a general algorithm design paradigm:
 - **Divide**: divide the input data S in two or more disjoint subsets S_1, S_2, \dots
 - **Recur**: solve the subproblems recursively
 - **Conquer**: combine the solutions for S_1, S_2, \dots , into a solution for S
- ◆ The base case for the recursion are subproblems of constant size
- ◆ Analysis can be done using **recurrence equations**



Merge-Sort Review

- ◆ Merge-sort on an input sequence S with n elements consists of three steps:
 - **Divide**: partition S into two sequences S_1 and S_2 of about $n/2$ elements each
 - **Recur**: recursively sort S_1 and S_2
 - **Conquer**: merge S_1 and S_2 into a unique sorted sequence

Algorithm *mergeSort*(S, C)

Input sequence S with n elements, comparator C

Output sequence S sorted according to C

if $S.size() > 1$

$(S_1, S_2) \leftarrow partition(S, n/2)$

mergeSort(S_1, C)

mergeSort(S_2, C)

$S \leftarrow merge(S_1, S_2)$

Recurrence Equation Analysis

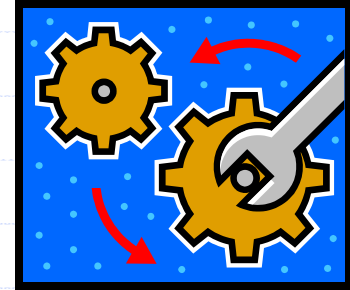


- ◆ The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes at most bn steps, for some constant b .
- ◆ Likewise, the basis case ($n < 2$) will take at b most steps.
- ◆ Therefore, if we let $T(n)$ denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

- ◆ We can therefore analyze the running time of merge-sort by finding a **closed form solution** to the above equation.
 - That is, a solution that has $T(n)$ only on the left-hand side.

Iterative Substitution

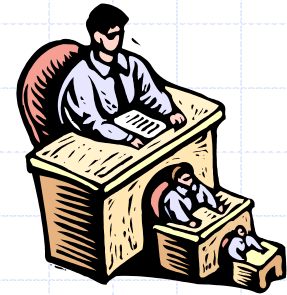


- ◆ In the iterative substitution, or “plug-and-chug,” technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern:

$$\begin{aligned}T(n) &= 2T(n/2) + bn \\ &= 2(2T(n/2^2)) + b(n/2) + bn \\ &= 2^2 T(n/2^2) + 2bn \\ &= 2^3 T(n/2^3) + 3bn \\ &= 2^4 T(n/2^4) + 4bn \\ &= \dots \\ &= 2^i T(n/2^i) + ibn\end{aligned}$$

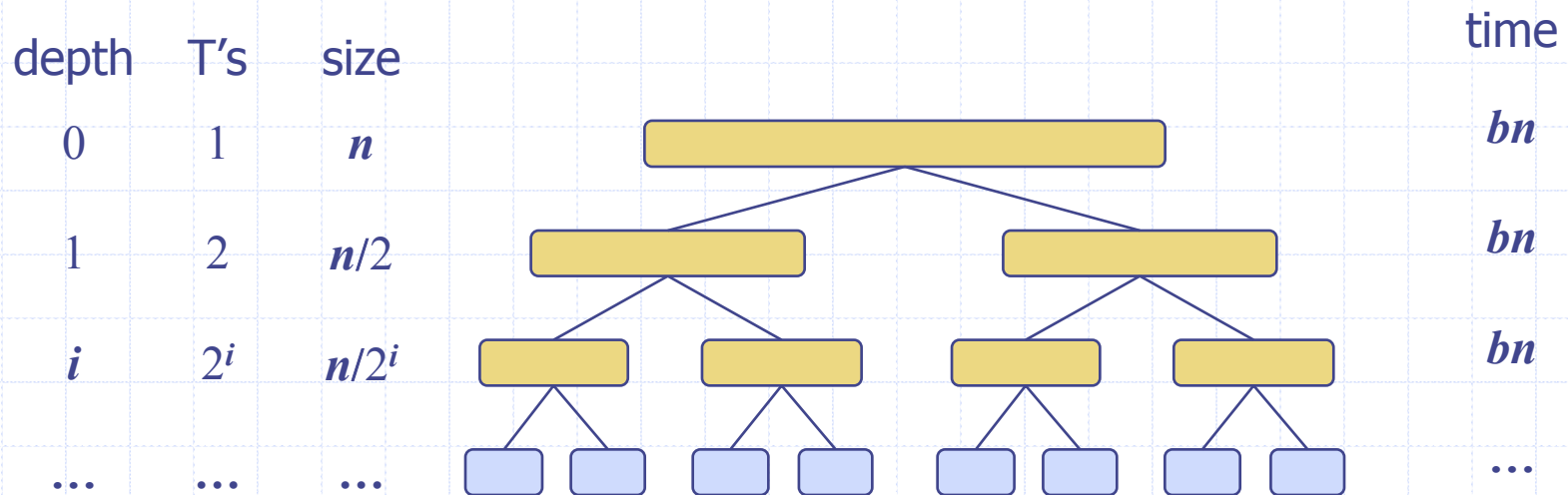
- ◆ Note that base, $T(n)=b$, case occurs when $2^i=n$. That is, $i = \log n$.
- ◆ So,
$$T(n) = bn + bn \log n$$
- ◆ Thus, $T(n)$ is $O(n \log n)$.

The Recursion Tree



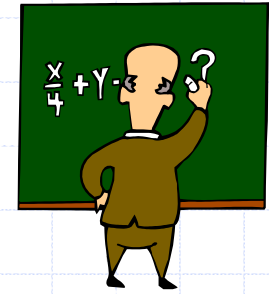
- ◆ Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$



Total time = $bn + bn \log n$
(last level plus all previous levels)

Guess-and-Test Method



- ◆ In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

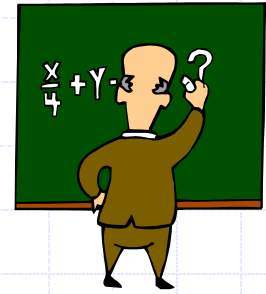
$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

- ◆ Guess: $T(n) < cn \log n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn \log n \\ &= 2(c(n/2) \log(n/2)) + bn \log n \\ &= cn(\log n - \log 2) + bn \log n \\ &= cn \log n - cn + bn \log n \end{aligned}$$

- ◆ Wrong: we cannot make this last line be less than $cn \log n$

Guess-and-Test Method, Part 2



- ◆ Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn \log n & \text{if } n \geq 2 \end{cases}$$

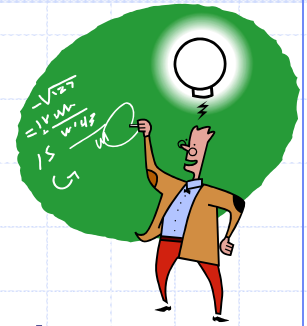
- ◆ Guess #2: $T(n) < cn \log^2 n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn \log n \\ &= 2(c(n/2) \log^2(n/2)) + bn \log n \\ &= cn(\log n - \log 2)^2 + bn \log n \\ &= cn \log^2 n - 2cn \log n + cn + bn \log n \\ &\leq cn \log^2 n \end{aligned}$$

- if $c > b$.

- ◆ So, $T(n)$ is $O(n \log^2 n)$.
- ◆ In general, to use this method, you need to have a good guess and you need to be good at induction proofs.

Master Method



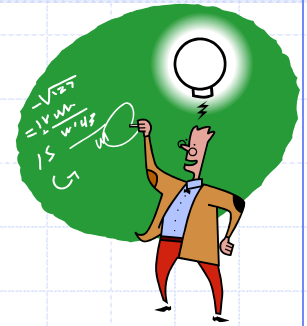
- ◆ Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

- ◆ The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Master Method, Example 1



◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

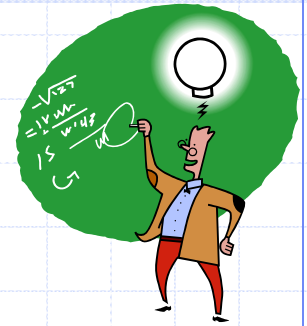
1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 4T(n/2) + n$$

Solution: $\log_b a = 2$, so case 1 says $T(n)$ is $O(n^2)$.

Master Method, Example 2



◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

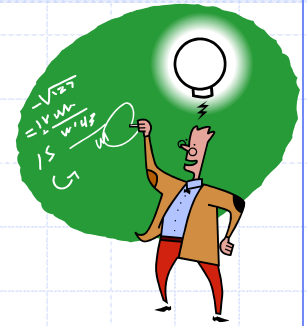
1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 2T(n/2) + n \log n$$

Solution: $\log_b a = 1$, so case 2 says $T(n)$ is $O(n \log^2 n)$.

Master Method, Example 3



◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

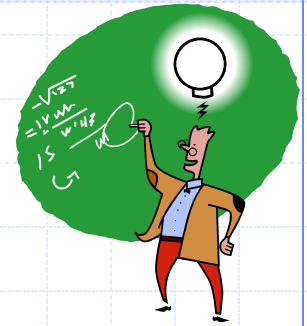
1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = T(n/3) + n \log n$$

Solution: $\log_b a = 0$, so case 3 says $T(n)$ is $O(n \log n)$.

Master Method, Example 4



◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

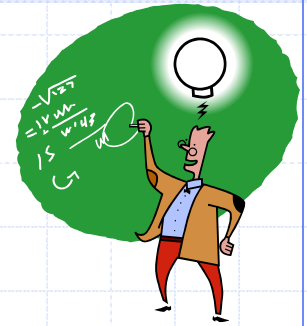
1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
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3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 8T(n/2) + n^2$$

Solution: $\log_b a = 3$, so case 1 says $T(n)$ is $O(n^3)$.

Master Method, Example 5



◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

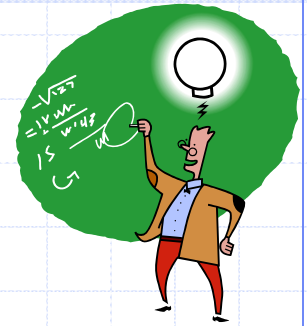
1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
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3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 9T(n/3) + n^3$$

Solution: $\log_b a = 2$, so case 3 says $T(n)$ is $O(n^3)$.

Master Method, Example 6



◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

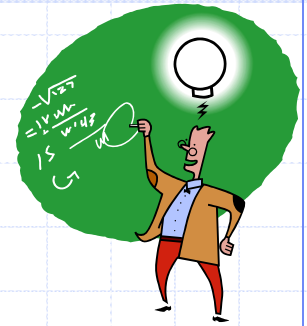
1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = T(n/2) + 1 \quad (\text{binary search})$$

Solution: $\log_b a = 0$, so case 2 says $T(n)$ is $O(\log n)$.

Master Method, Example 7



◆ The form:
$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

◆ The Master Theorem:

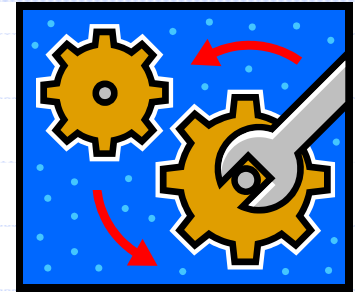
1. if $f(n)$ is $O(n^{\log_b a - \epsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \epsilon})$, then $T(n)$ is $\Theta(f(n))$,
provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

◆ Example:

$$T(n) = 2T(n/2) + \log n \quad (\text{heap construction})$$

Solution: $\log_b a = 1$, so case 1 says $T(n)$ is $O(n)$.

Iterative “Proof” of the Master Theorem

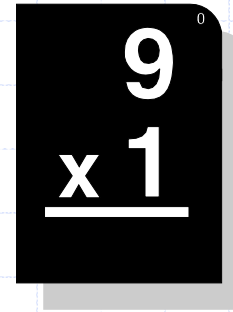


- ◆ Using iterative substitution, let us see if we can find a pattern:

$$\begin{aligned}T(n) &= aT(n/b) + f(n) \\&= a(aT(n/b^2)) + f(n/b) + bn \\&= a^2T(n/b^2) + af(n/b) + f(n) \\&= a^3T(n/b^3) + a^2f(n/b^2) + af(n/b) + f(n) \\&= \dots \\&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)\end{aligned}$$

- ◆ We then distinguish the three cases as
 - The first term is dominant
 - Each part of the summation is equally dominant
 - The summation is a geometric series

Integer Multiplication



Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

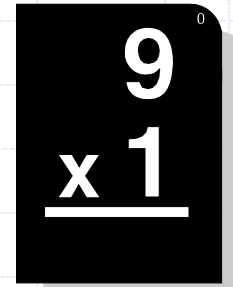
$$J = J_h 2^{n/2} + J_l$$

- We can then define $I * J$ by multiplying the parts and adding:

$$\begin{aligned} I * J &= (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l) \\ &= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \end{aligned}$$

- So, $T(n) = 4T(n/2) + n$, which implies $T(n)$ is $O(n^2)$.
- But that is no better than the algorithm we learned in grade school.

An Improved Integer Multiplication Algorithm



Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- Observe that there is a different way to multiply parts:

$$\begin{aligned} I * J &= I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l \end{aligned}$$

- So, $T(n) = 3T(n/2) + n$, which implies $T(n)$ is $O(n^{\log_2 3})$, by the Master Theorem.
- Thus, $T(n)$ is $O(n^{1.585})$.